

# Casimir densities for a plate in de Sitter spacetime

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## Abstract

Wightman function, the vacuum expectation values of the field squared and the energy-momentum tensor are investigated for a scalar field with general curvature coupling parameter in the geometry of a plate in the de Sitter spacetime. Robin boundary condition for the field operator is assumed on the plate. The vacuum expectation values are presented as the sum of two terms. The first one corresponds to the geometry of de Sitter bulk without boundaries and the second one is induced by the presence of the plate. We show that for non-conformal fields the vacuum energy-momentum tensor is non-diagonal with the off-diagonal component corresponding to the energy flux along the direction perpendicular to the plate. In dependence of the parameters, this flux can be either positive or negative. The asymptotic behavior of the field squared, vacuum energy density and stresses near the plate and at large distances is investigated.

## 1 Introduction

De Sitter (dS) spacetime is one of the simplest and most interesting spacetimes allowed by general relativity. Quantum field theory in this background has been extensively studied during the past two decades. Much of early interest was motivated by the questions related to the quantization of fields propagating on curved backgrounds. dS spacetime has a high degree of symmetry and numerous physical problems are exactly solvable on this background. The importance of this theoretical work increased by the appearance of the inflationary cosmology scenario [1]. In most inflationary models, an approximately dS spacetime is employed to solve a number of problems in standard cosmology. During an inflationary epoch, quantum fluctuations in the inflaton field introduce inhomogeneities which play a central role in the generation of cosmic structures from inflation. More recently, astronomical observations of high redshift supernovae, galaxy clusters and cosmic microwave background [2] indicate that at the present epoch the Universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the Universe would accelerate indefinitely, the standard cosmology would lead to an asymptotic dS universe. Hence, the investigation of physical effects in dS spacetime is important for understanding both the early Universe and its future. Another motivation for investigations of dS based quantum theories is related to the holographic duality between quantum gravity on dS spacetime and a quantum field theory living on boundary identified with the timelike infinity of dS spacetime [3].

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The Casimir effect is among the most striking macroscopic manifestations of non-trivial properties of the quantum vacuum. The presence of reflecting boundaries alters the zero-point modes of a quantized field and shifts the vacuum expectation values of quantities, such as the energy density and stresses. As a result, vacuum forces arise acting on constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold, the boundary geometry and the specific boundary conditions imposed on the field. Since the original work by Casimir [4] many theoretical and experimental works have been done on this problem (see, e.g., [5]–[11] and references therein).

The Casimir effect can be viewed as a polarization of vacuum by boundary conditions. The interaction of fluctuating quantum fields with background gravitational fields gives rise to another type of vacuum polarization (see, for instance, [12, 13]). Here we will study an example with both types of vacuum polarization. Namely, we evaluate the vacuum expectation values for the field squared and the energy-momentum tensor of a scalar field with general curvature coupling parameter induced by a plane boundary on background of  $(D + 1)$ -dimensional dS spacetime. Previously the Casimir effect on background of dS spacetime described in planar coordinates was investigated in Refs. [14] for a conformally coupled massless scalar field. In this case the problem is conformally related to the corresponding problem in Minkowski spacetime and the vacuum characteristics are generated from those for the Minkowski counterpart multiplying by the conformal factor. In particular, for the geometry of a single plate the vacuum expectation value of the energy-momentum tensor vanishes. In [15] the vacuum expectation value of the energy-momentum tensor for a conformally coupled scalar field is investigated in dS spacetime with static coordinates in presence of curved branes on which the field obeys the Robin boundary conditions with coordinate dependent coefficients (for investigations of the Casimir energy in braneworld models with dS branes see Refs. [16]). In these papers the conformal relation between dS and Rindler spacetimes and the results for the Rindler counterpart [17] were used. More recently, the Casimir densities in dS spacetime with toroidally compactified spatial dimensions were investigated in [18].

The present paper is organized as follows. In the next section the geometry of the problem is described and the corresponding positive frequency Wightman function is evaluated for the general case of the curvature coupling parameter and for Robin boundary condition on the plane boundary. In quantum field theory on curved backgrounds among the important quantities describing the local properties of a quantum field and quantum back-reaction effects are the expectation values of the field squared and the energy-momentum tensor. In sections 3 and 4, by using the expression for the Wightman function, we investigate the parts in these expectation values induced by the plate. Simple asymptotic formulae are obtained for small and large proper distances from the plate with respect to the dS curvature scale. The special cases of Dirichlet and Neumann boundary conditions are discussed in section 5. The main results are summarized in section 6.

## 2 Wightman function

For a scalar field with curvature coupling parameter  $\xi$  the field equation has the form

$$\left(\nabla_l \nabla^l + m^2 + \xi R\right) \varphi = 0, \quad (1)$$

where  $\nabla_l$  is the covariant derivative operator and  $R$  is the Ricci scalar for the background spacetime. The values of the curvature coupling parameter  $\xi = 0$  and  $\xi = \xi_D \equiv (D - 1)/4D$ , with  $D$  being the number of spatial dimensions, correspond to the most important special cases of minimally and conformally coupled fields. In the present paper the background geometry is

the  $(D+1)$ -dimensional de Sitter spacetime. We write the corresponding line element in planar coordinates most appropriate for cosmological applications:

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{i=1}^D (dz^i)^2. \quad (2)$$

The Ricci scalar and the corresponding cosmological constant are related to the parameter  $\alpha$  in the scale factor by the formulae

$$R = D(D+1)/\alpha^2, \quad \Lambda = D(D-1)/2\alpha^2. \quad (3)$$

In addition to the synchronous time coordinate  $t$  it is convenient to introduce the conformal time in accordance with  $\tau = -\alpha e^{-t/\alpha}$ ,  $-\infty < \tau < 0$ . In terms of this coordinate the line element takes the conformally flat form:

$$ds^2 = \alpha^2 \tau^{-2} [d\tau^2 - \sum_{i=1}^D (dz^i)^2]. \quad (4)$$

We will assume the presence of a plate located at  $z^D = 0$  on which the field obeys the Robin boundary condition

$$(1 + \beta \partial_{z^D}) \varphi = 0, \quad z^D = 0, \quad (5)$$

with a constant coefficient  $\beta$ . Robin type conditions are an extension of Dirichlet and Neumann boundary conditions and appear in a variety of situations, including the considerations of vacuum effects for a confined charged scalar field in external fields, spinor and gauge field theories, quantum gravity and supergravity. Robin boundary conditions with coefficients related to the curvature scale of the background spacetime naturally arise for scalar and fermion bulk fields in braneworld models. In this paper we are interested in the boundary induced effects on the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor assuming that the field is prepared in the Bunch-Davies vacuum state [19]. These VEVs are obtained from the corresponding Wightman function in the coincidence limit of the arguments. In addition, the Wightman function determines the response of the particle detector of Unruh-DeWitt type, moving through the vacuum under consideration (see, for instance, [12]).

In order to evaluate the Wightman function we employ the mode-sum formula

$$W(x, x') = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \sum_{\sigma} \varphi_{\sigma}(x) \varphi_{\sigma}^*(x'), \quad (6)$$

where  $\{\varphi_{\sigma}(x), \varphi_{\sigma}^*(x)\}$  is a complete set of solutions to the classical field equation satisfying the boundary condition (5). The eigenfunctions realizing the Bunch-Davies vacuum state and obeying the boundary condition have the form

$$\varphi_{\sigma}(x) = C_{\sigma} \eta^{D/2} H_{\nu}^{(1)}(K\eta) \cos[k_D z^D + \alpha(k_D)] e^{i\mathbf{k} \cdot \mathbf{z}}, \quad \sigma = (\mathbf{k}, k_D), \quad (7)$$

with the notations  $\eta = |\tau|$ ,  $K = \sqrt{k^2 + k_D^2}$ ,

$$\mathbf{k} = (k_1, \dots, k_{D-1}), \quad \mathbf{z} = (z^1, \dots, z^{D-1}). \quad (8)$$

The order of the Hankel function  $H_{\nu}^{(1)}(x)$  in (7) is given by the expression

$$\nu = [D^2/4 - D(D+1)\xi - m^2\alpha^2]^{1/2}, \quad (9)$$

and the function  $\alpha(k_D)$  is defined by the relation

$$e^{2i\alpha(k_D)} = \frac{i\beta k_D - 1}{i\beta k_D + 1}. \quad (10)$$

With this definition the function (7) satisfies the boundary condition at  $z^D = 0$ . Note that we also have the formulae

$$\cos[\alpha(k_D)] = \frac{k_D\beta}{\sqrt{1 + k_D^2\beta^2}}, \quad \sin[\alpha(k_D)] = \frac{1}{\sqrt{1 + k_D^2\beta^2}}. \quad (11)$$

The coefficient  $C_\sigma$  in the expression for the eigenfunctions is found from the orthonormalization condition

$$-i \int d\mathbf{z} \int_0^\infty dz^D \sqrt{|g|} g^{00} [\varphi_\sigma(x) \partial_\tau \varphi_{\sigma'}^*(x) - \varphi_{\sigma'}^*(x) \partial_\tau \varphi_\sigma(x)] = \delta(\mathbf{k} - \mathbf{k}') \delta(k_D - k'_D). \quad (12)$$

By using the Wronskian relation for the Hankel functions, it can be seen that

$$H_\nu^{(1)}(K\eta) H_{\nu^*}^{(2)'}(K\eta) - H_{\nu^*}^{(2)}(K\eta) H_\nu^{(1)'}(K\eta) = -\frac{4ie^{-(\nu-\nu^*)\pi i/2}}{\pi K\eta}. \quad (13)$$

In addition, by making use of the definition of the function  $\alpha(k_D)$ , we can derive the integration formula

$$\int_0^\infty dz^D \cos[k_D z^D + \alpha(k_D)] \cos[k'_D z^D + \alpha(k'_D)] = \frac{\pi}{2} \delta(k_D - k'_D). \quad (14)$$

Combining the last two relations with (12), for the normalization coefficient we find

$$C_\sigma^2 = \frac{e^{(\nu-\nu^*)\pi i/2}}{2(2\pi\alpha)^{D-1}}. \quad (15)$$

Substituting the eigenfunctions (7) with the normalization coefficient (15) into the mode-sum formula (6) for the Wightman function, one finds

$$\begin{aligned} W(x, x') &= \frac{8(\eta\eta')^{D/2}}{(2\pi)^{D+1}\alpha^{D-1}} \int d\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{z}-\mathbf{z}')} \int_0^\infty du \cos[uz^D + \alpha(u)] \cos[uz^{D'} + \alpha(u)] \\ &\quad \times K_\nu(\eta\sqrt{k^2 + u^2}e^{-\pi i/2}) K_\nu(\eta'\sqrt{k^2 + u^2}e^{\pi i/2}). \end{aligned} \quad (16)$$

where we have written the Hankel functions in terms of the modified Bessel function  $K_\nu(z)$ . The formula (16) can also be presented in an equivalent form

$$\begin{aligned} W(x, x') &= W_{\text{dS}}(x, x') + \frac{4(\eta\eta')^{D/2}}{(2\pi)^{D+1}\alpha^{D-1}} \int d\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{z}-\mathbf{z}')} \int_0^\infty du \cos[u(z^D + z^{D'}) + 2\alpha(u)] \\ &\quad \times K_\nu(\eta\sqrt{k^2 + u^2}e^{-\pi i/2}) K_\nu(\eta'\sqrt{k^2 + u^2}e^{\pi i/2}), \end{aligned} \quad (17)$$

where

$$\begin{aligned} W_{\text{dS}}(x, x') &= \frac{4(\eta\eta')^{D/2}}{(2\pi)^{D+1}\alpha^{D-1}} \int d\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{z}-\mathbf{z}')} \int_0^\infty du \cos[u(z^D - z^{D'})] \\ &\quad \times K_\nu(\eta\sqrt{k^2 + u^2}e^{-\pi i/2}) K_\nu(\eta'\sqrt{k^2 + u^2}e^{\pi i/2}), \end{aligned} \quad (18)$$

is the Wightman function for the dS spacetime without boundaries and the second term on the right is induced by the plate at  $z^D = 0$ . Two-point function (18) is well investigated in the literature [19, 20, 21, 22, 23] (see also [12]) and can be presented in the form

$$W_{\text{dS}}(x, x') = \frac{\alpha^{1-D} \Gamma(D/2 + \nu) \Gamma(D/2 - \nu)}{2^{(D+3)/2} \pi^{(D+1)/2} (u^2 - 1)^{(D-1)/4}} P_{\nu-1/2}^{(1-D)/2}(u), \quad (19)$$

where  $P_\nu^\mu(u)$  is the associated Legendre function of the first kind and

$$u = -1 + \frac{\sum_{l=1}^D (z^l - z'^l)^2 - (\eta - \eta')^2}{2\eta\eta'}. \quad (20)$$

An alternative form is obtained by using the relation between the associated Legendre function and the hypergeometric function (see, for example, [24]).

In order to transform the boundary induced part in (17) into a more convenient form we write

$$2 \cos[u(z^D + z^{D'}) + 2\alpha(u)] = e^{iu(z^D + z^{D'})} \frac{i\beta u - 1}{i\beta u + 1} + e^{-iu(z^D + z^{D'})} \frac{i\beta u + 1}{i\beta u - 1}. \quad (21)$$

Now, in the boundary induced part of (17) we rotate the integration contour over  $u$  by the angles  $\pi/2$  and  $-\pi/2$  for the integrals with the first and second terms on the right-hand side of (21), respectively. Assuming that  $\beta \leq 0$ , the Wightman function is presented in the form

$$\begin{aligned} W(x, x') &= W_{\text{dS}}(x, x') + \frac{(\eta\eta')^{D/2}}{2(2\pi)^D \alpha^{D-1}} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{z} - \mathbf{z}')} \int_k^\infty du e^{-u(z^D + z^{D'})} \frac{\beta u + 1}{\beta u - 1} \\ &\times \{K_\nu(\eta y) [I_{-\nu}(\eta' y) + I_\nu(\eta' y)] + [I_\nu(\eta y) + I_{-\nu}(\eta y)] K_\nu(\eta' y)\}_{y=\sqrt{u^2 - k^2}}, \quad (22) \end{aligned}$$

where  $I_\nu(z)$  is the modified Bessel function of the first kind. In deriving this formula we have used the relations  $K_\nu(z e^{\pm \pi i}) = e^{\mp \nu \pi i} K_\nu(z) \mp \pi i I_\nu(z)$ . In the case  $\beta > 0$  the first and second terms on the right of (21) have simple poles at the points  $i/\beta$  and  $-i/\beta$  respectively. After the rotation of the integration contour these points should be avoided by small semicircles in the right half-plane. The integrals along these semicircles give additional contributions which should be added to the expression (22) where now the integration over  $u$  is understood in the sense of the principal value. In order to not complicate formulae, in the discussion below we will consider the case  $\beta \leq 0$ . Note that in (22) the integration over the angular part of the vector  $\mathbf{k}$  can be done explicitly and the expression for the Wightman function takes the form

$$\begin{aligned} W(x, x') &= W_{\text{dS}}(x, x') + \frac{(\eta\eta')^{D/2}}{2(2\pi)^{(D+1)/2} \alpha^{D-1}} \int_0^\infty dy y \\ &\times \{K_\nu(\eta y) [I_{-\nu}(\eta' y) + I_\nu(\eta' y)] + [I_\nu(\eta y) + I_{-\nu}(\eta y)] K_\nu(\eta' y)\} \\ &\times \int_0^\infty dk k^{D-2} \frac{J_{(D-3)/2}(k|\mathbf{z} - \mathbf{z}'|)}{(k|\mathbf{z} - \mathbf{z}'|)^{(D-3)/2}} \frac{e^{-u(z^D + z^{D'})}}{u} \frac{\beta u + 1}{\beta u - 1} \Big|_{u=\sqrt{y^2 + k^2}}, \quad (23) \end{aligned}$$

where  $J_\nu(z)$  is the Bessel function of the first kind.

For Dirichlet and Neumann boundary condition one has  $\beta = 0$  and  $1/\beta = 0$ , respectively, and the integral over  $k$  in (23) is evaluated by making use of the formula [25]

$$\int_0^\infty dk k^{D-2} \frac{J_{(D-3)/2}(k|\mathbf{z} - \mathbf{z}'|)}{(k|\mathbf{z} - \mathbf{z}'|)^{(D-3)/2}} \frac{e^{-\sqrt{y^2 + k^2}(z^D + z^{D'})}}{\sqrt{y^2 + k^2}} = \sqrt{\frac{2}{\pi}} y^{D/2-1} \frac{K_{D/2-1}(y\lambda)}{\lambda^{D/2-1}}, \quad (24)$$

with the notation  $\lambda = \sqrt{|\mathbf{z} - \mathbf{z}'|^2 + (z^D + z'^D)^2}$ . For the corresponding Wightman functions we find

$$W^{(J)}(x, x') = W_{\text{dS}}(x, x') - \delta^{(J)} \frac{(\eta\eta')^{D/2} \alpha^{1-D}}{2(2\pi)^{D/2+1} \lambda^{D/2-1}} \int_0^\infty dy y^{D/2} K_{D/2-1}(y\lambda) \\ \times \{K_\nu(\eta y) [I_{-\nu}(\eta' y) + I_\nu(\eta' y)] + [I_\nu(\eta y) + I_{-\nu}(\eta y)] K_\nu(\eta' y)\}, \quad (25)$$

where  $J=D$  and  $J=N$  for Dirichlet and Neumann boundary conditions and  $\delta^{(D)} = 1$ ,  $\delta^{(N)} = -1$ .

### 3 VEV of the field squared

The VEV of the field squared is obtained from the Wightman function taking the coincidence limit of the arguments. This limit is divergent and some renormalization procedure is necessary. The important point here is that for points away from the plate the divergences are contained in the part corresponding to the boundary-free dS spacetime and the boundary induced part is finite. As we have already extracted the first part, the renormalization procedure is reduced to the renormalization of the boundary-free dS part which is already done in the literature (see, for instance, [19, 20, 21]). As a result the renormalized VEV of the field squared is presented in the form

$$\langle \varphi^2 \rangle = \langle \varphi^2 \rangle_{\text{dS}} + \langle \varphi^2 \rangle_{\text{pl}}, \quad (26)$$

where  $\langle \varphi^2 \rangle_{\text{dS}}$  is the VEV in dS spacetime without boundaries and the part

$$\langle \varphi^2 \rangle_{\text{pl}} = \frac{4(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2) \alpha^{D-1}} \int_0^\infty dy y [I_\nu(y) + I_{-\nu}(y)] K_\nu(y) \int_y^\infty dx G(x, y), \quad (27)$$

is induced by the plate. Here and in the discussion below we use the notation

$$G(x, y) = (x^2 - y^2)^{(D-3)/2} e^{-2z^D x/\eta} \frac{\beta x/\eta + 1}{\beta x/\eta - 1}. \quad (28)$$

Due to the maximal symmetry of the dS spacetime the VEV  $\langle \varphi^2 \rangle_{\text{dS}}$  does not depend on the spacetime point. Note that the boundary induced VEV is a function on the combinations  $z^D/\eta$  and  $\beta/\eta$  only. The ratio  $z^D/\eta$  is the proper distance from the plate measured in units of the curvature scale  $\alpha$ .

For a conformally coupled massless scalar field we have  $\xi = \xi_D$ ,  $\nu = 1/2$  and the modified Bessel functions in (27) are expressed in terms of the elementary functions with the result  $[I_\nu(y) + I_{-\nu}(y)] K_\nu(y) = 1/y$ . By making use of the formula

$$\int_0^\infty dy \int_y^\infty dx (x^2 - y^2)^{(D-3)/2} f(x) = \frac{\sqrt{\pi} \Gamma((D-1)/2)}{2\Gamma(D/2)} \int_0^\infty dr r^{D-2} f(r), \quad (29)$$

the boundary induced part in the VEV of the field squared is presented in the form

$$\langle \varphi^2 \rangle_{\text{pl}} = \frac{(\eta/\alpha)^{D-1}}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^\infty dx x^{D-2} e^{-2xz^D} \frac{\beta x + 1}{\beta x - 1}. \quad (30)$$

Formula (29) is obtained changing the integration variable to  $u = \sqrt{x^2 - y^2}$  and introducing polar coordinates in the plane  $(y, u)$ . Note that the result (30) could also be directly obtained by using the fact that in the special case under consideration the problem is conformally related to the corresponding problem for a Robin plate in the Minkowski spacetime [26, 27]. From this

relation it follows that  $\langle \varphi^2 \rangle_{\text{pl}} = [a(\eta)]^{1-D} \langle \varphi^2 \rangle_{\text{pl}}^{(\text{M})}$ , with the scale factor  $a(\eta) = \alpha/\eta$ , which leads to the result (30).

Now let us investigate the behavior of the boundary induced VEV  $\langle \varphi^2 \rangle_{\text{pl}}$  in the asymptotic regions of the ratio  $z^D/\eta$ . For small values of this ratio,  $z^D/\eta \ll 1$ , which correspond to small proper distances from the plate with respect to the curvature scale  $\alpha$ , we introduce new integration variables  $z = xz^D/\eta$  and  $u = yz^D/\eta$ . The argument of the modified Bessel functions becomes  $u\eta/z^D$  and is large. By using the asymptotic relation  $[I_\nu(y) + I_{-\nu}(y)] K_\nu(y) \approx 1/y$  for large values  $y$  and fixed  $\nu$ , we find

$$\langle \varphi^2 \rangle_{\text{pl}} \approx \frac{(\alpha z^D/\eta)^{1-D}}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^\infty dx x^{D-2} e^{-2x} \frac{\beta x/z^D + 1}{\beta x/z^D - 1}, \quad z^D/\eta \ll 1. \quad (31)$$

Hence, in the limit under consideration to the leading order the VEV coincides with the corresponding result for a conformally coupled massless scalar. For fixed value  $z^D$  this limit corresponds to early stages of the cosmological expansion. As the boundary-free part  $\langle \varphi^2 \rangle_{\text{ds}}$  is a constant, we conclude that at small proper distances from the plate the total VEV of the field squared is dominated by the boundary induced part. From (31) it follows that for Dirichlet boundary condition ( $\beta = 0$ ) this part is negative. For non-Dirichlet boundary conditions, assuming that  $z^D \ll |\beta|$ , near the plate the VEV of the field squared is positive.

In the opposite limit of large values for the ratio  $z^D/\eta$  (large proper distances from the plate with respect to the curvature scale  $\alpha$ ), the main contribution in the integral over  $x$  in (27) comes from the region near the lower limit of the integration and for positive values of the parameter  $\nu$  to the leading order one has

$$\langle \varphi^2 \rangle_{\text{pl}} \approx \frac{\delta_\beta \Gamma((D+1)/2 - 2\nu)}{2^{1-2\nu} (2\pi)^{(D+1)/2} \alpha^{D-1}} \frac{\Gamma(\nu)}{\Gamma(1-\nu)} \left( \frac{\eta}{2z^D} \right)^{D-2\nu}, \quad z^D/\eta \gg 1, \quad (32)$$

where  $\delta_\beta = 1$  for  $1/\beta = 0$  and  $\delta_\beta = -1$  for  $1/\beta \neq 0$ . In the case  $\nu = 0$  the VEV of the field squared behaves as  $\langle \varphi^2 \rangle_{\text{pl}} \propto (\eta/z^D)^D \ln(z^D/\eta)$ . For fixed values of  $z^D$  this limit corresponds to late stages of the cosmological expansion. For imaginary values of  $\nu$ , by using the relation

$$K_\nu(y) [I_{-\nu}(y) + I_\nu(y)] \approx \frac{\pi}{\sinh(|\nu|\pi)} \text{Im} \left[ \frac{(y/2)^{-2i|\nu|}}{\Gamma^2(1-i|\nu|)} \right], \quad (33)$$

for small values of  $y$ , in the leading order we find

$$\langle \varphi^2 \rangle_{\text{pl}} \approx \frac{\delta_\beta (2\pi)^{-(D+1)/2}}{\alpha^{D-1} (2z^D/\eta)^D} \frac{\pi B_0}{\sinh(|\nu|\pi)} \sin[2|\nu| \ln(2z^D/\eta) + \phi_0]. \quad (34)$$

In this formula the constants  $B_0$  and  $\phi_0$  are defined by the relation

$$B_0 e^{i\phi_0} = \frac{2^{2i|\nu|} \Gamma((D+1)/2 - 2i|\nu|)}{\Gamma^2(1-i|\nu|)}.$$

As we see, in this case the behavior of the boundary induced VEV as a function of the proper distance from the plate is oscillatory damping. Note that in terms of the synchronous time coordinate the expression in the phase of (34) is written in the form  $\ln(2z^D/\eta) = t/\alpha + \ln(2z^D/\alpha)$ . From the asymptotic expressions given above we conclude that at large proper distances from the plate the VEV of the field squared is dominated by the boundary-free part  $\langle \varphi^2 \rangle_{\text{ds}}$ .

## 4 Energy-momentum tensor

Now we turn to the investigation of the VEV for the energy-momentum tensor. Having the Wightman function we can evaluate this VEV by making use of the formula

$$\langle T_{ik} \rangle = \lim_{x' \rightarrow x} \partial_i \partial'_k W(x, x') + \left[ (\xi - 1/4) g_{ik} \nabla_l \nabla^l - \xi \nabla_i \nabla_k - \xi R_{ik} \right] \langle \varphi^2 \rangle, \quad (35)$$

where  $R_{ik} = Dg_{ik}/\alpha^2$  is the Ricci tensor for the dS spacetime. Similar to the case of the field squared, the VEV of the energy-momentum tensor is presented in the decomposed form

$$\langle T_{ik} \rangle = \langle T_{ik} \rangle_{\text{dS}} + \langle T_{ik} \rangle_{\text{pl}}, \quad (36)$$

where  $\langle T_{ik} \rangle_{\text{dS}}$  is the corresponding VEV in dS spacetime without boundaries and  $\langle T_{ik} \rangle_{\text{pl}}$  is induced by the plate. For points away the plate the latter is finite and the renormalization is necessary for the boundary-free part only. Due to the dS invariance of the Bunch-Davies vacuum this part is proportional to the metric tensor with a constant coefficient and is well investigated in the literature [19, 20, 21]. For this reason in the discussion below we will be focused on the boundary induced part.

In order to evaluate the boundary induced part in the VEV of the energy-momentum tensor we need the expression for the covariant d'Alembertian acted on the VEV of the field squared:

$$\begin{aligned} \nabla_l \nabla^l \langle \varphi^2 \rangle_{\text{pl}} &= \frac{4(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2)\alpha^{D+1}} \int_0^\infty dy y^{3-D} \int_y^\infty dx G(x, y) \\ &\times \left( \partial_y^2 + \frac{1-D}{y} \partial_y - \frac{4x^2}{y^2} \right) \tilde{I}_\nu(y) \tilde{K}_\nu(y), \end{aligned} \quad (37)$$

where the notations

$$\tilde{K}_\nu(z) = z^{D/2} K_\nu(z), \quad \tilde{I}_\nu(z) = z^{D/2} [I_\nu(z) + I_{-\nu}(z)] \quad (38)$$

are introduced. Now, by using the formula (35), after long calculations, we present the VEV of the energy-momentum tensor in the form

$$\langle T_k^i \rangle_{\text{pl}} = \frac{4(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2)\alpha^{D+1}} \int_0^\infty dy y^{1-D} \int_y^\infty dx G(x, y) \left[ F_k^i(y) + x^2 F_k^i \tilde{I}_\nu(y) \tilde{K}_\nu(y) \right], \quad (39)$$

where the function  $G(x, y)$  is defined by (28). In formula (39) we have introduced the notations (no summation over  $i$ )

$$\begin{aligned} F_0^0(y) &= y^2 \tilde{I}'_\nu(y) \tilde{K}'_\nu(y) - \left[ \frac{1}{4} y^2 \partial_y^2 + D(\xi - \xi_D) y \partial_y + \xi D \right] \tilde{I}_\nu(y) \tilde{K}_\nu(y), \\ F_i^i(y) &= F_D^D(y) + \frac{y^2}{D-1} \tilde{I}_\nu(y) \tilde{K}_\nu(y), \quad i = 1, 2, \dots, D-1, \\ F_D^D(y) &= \left\{ \left( \xi - \frac{1}{4} \right) y^2 \partial_y^2 + \left[ \xi(2-D) + \frac{D-1}{4} \right] y \partial_y - \xi D \right\} \tilde{I}_\nu(y) \tilde{K}_\nu(y), \\ F_0^D(y) &= 2x [(\xi - 1/4) y \partial_y + \xi] \tilde{I}_\nu(y) \tilde{K}_\nu(y), \end{aligned} \quad (40)$$

and

$$F_0^0 = 1 - 4\xi, \quad F_i^i = 1 - 4\xi - \frac{1}{D-1}, \quad F_0^D = F_D^D = 0. \quad (41)$$



As in the case of the field squared, the VEVs of the energy-momentum tensor are functions of the ratios  $z^D/\eta$  and  $\beta/\eta$  only. Note that, by using the well-known properties of the modified Bessel functions, the function  $F_0^0(y)$  can also be written in the form

$$F_0^0(y) = [(y^2/4)\partial_y^2 - D(\xi + \xi_D)y\partial_y - y^2 + D^2\xi + m^2\alpha^2] \tilde{I}_\nu(y)\tilde{K}_\nu(y). \quad (42)$$

As we see, the vacuum energy-momentum tensor is non-diagonal with the off-diagonal component  $\langle T_0^D \rangle_{\text{pl}}$  which describes the energy flux along the direction normal to the plate. This flux can be either positive or negative (see examples plotted in figures below for Dirichlet boundary condition). Note that this type of the energy flux appears also in the geometry of a cosmic string on background of Friedmann-Robertson-Walker and dS spacetimes [28, 29].

It can be checked that the boundary induced parts in the VEV of the energy-momentum tensor satisfy the trace relation

$$\langle T_i^i \rangle_{\text{pl}} = [D(\xi - \xi_D)\nabla_i\nabla^i + m^2] \langle \varphi^2 \rangle_{\text{pl}}. \quad (43)$$

In particular, the plate induced part is traceless for a conformally coupled massless scalar field. Trace anomaly is contained in the boundary-free part only.

For a conformally coupled massless field we have  $\nu = 1/2$ . In this case  $\tilde{I}_\nu(y)\tilde{K}_\nu(y) = y^{D-1}$ , and it can be checked that  $F_0^D(y) = F_D^D(y) = 0$ . For the energy density the corresponding expression takes the form

$$\begin{aligned} \langle T_0^0 \rangle_{\text{pl}} &= \frac{4(4\pi)^{-(D+1)/2}\eta^{D+1}}{D\Gamma((D-1)/2)\alpha^{D+1}} \int_0^\infty dy \int_y^\infty dx (x^2 - y^2)^{(D-3)/2} \\ &\quad \times e^{-2zx} \frac{\beta x + 1}{\beta x - 1} (x^2 - Dy^2), \end{aligned} \quad (44)$$

and for the stresses along the directions parallel to the plate we have (no summation over  $i$ )  $\langle T_i^i \rangle_{\text{pl}} = -(D-1)\langle T_0^0 \rangle_{\text{pl}}$ ,  $i = 1, 2, \dots, D-1$ . Introducing in (44) a new integration variable  $z = \sqrt{x^2 - y^2}$  and polar coordinates  $y = r \cos \theta$ ,  $z = r \sin \theta$ , we can see that the integral over  $\theta$  vanishes. Hence, for a conformally coupled field the plate induced part vanishes. This result can also be obtained by using the conformal relation between the problem under consideration and the corresponding problem in the Minkowski spacetime. In the latter geometry the Casimir energy-momentum tensor for a conformally coupled massless scalar field vanishes.

The general formulae for the VEV of the energy-momentum tensor are simplified in the asymptotic regions of the ratio  $z^D/\eta$ . For small values of this parameter, corresponding to small proper distances from the plate, by using the asymptotic formulae for the modified Bessel functions for large values of the argument, we find the following asymptotic behavior:

$$\begin{aligned} \langle T_l^l \rangle_{\text{pl}} &\approx -\frac{4(\xi - \xi_D)}{(\alpha z^D/\eta)^{D+1}} \mathcal{I}_D(\beta/z^D), \quad l = 0, 1, \dots, D-1, \\ \langle T_0^D \rangle_{\text{pl}} &\approx \frac{2D(\xi - \xi_D)}{\alpha(\alpha z^D/\eta)^D} \mathcal{I}_{D-1}(\beta/z^D), \quad \langle T_D^D \rangle_{\text{pl}} \approx -\frac{D(\xi - \xi_D)}{\alpha^2(\alpha z^D/\eta)^{D-1}} \mathcal{I}_{D-2}(\beta/z^D), \end{aligned} \quad (45)$$

where we have introduced the notation

$$\mathcal{I}_\mu(z) = \frac{(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dx x^\mu e^{-2x} \frac{zx + 1}{zx - 1}.$$

Hence, the VEVs diverge on the boundary. The ratio  $\alpha z^D/\eta$  in these formulae is the proper distance from the plate and the leading divergences in the components  $\langle T_l^l \rangle_{\text{pl}}$ ,  $l = 0, 1, \dots, D-1$ ,

are the same as those in the corresponding problem on the Minkowski bulk. For the latter problem the components  $\langle T_0^D \rangle_{\text{pl}}$  and  $\langle T_D^D \rangle_{\text{pl}}$  vanish. In the case of the dS bulk near the plate we have  $|\langle T_D^D \rangle_{\text{pl}}| \ll |\langle T_0^D \rangle_{\text{pl}}| \ll |\langle T_0^0 \rangle_{\text{pl}}|$ . Note that for a conformally coupled field the leading terms vanish and the next terms in the corresponding asymptotic expansions should be taken into account. The limit under consideration corresponds to points near the plate or to late stages of the cosmological expansion. In this limit the total VEV of the energy-momentum tensor is dominated by the boundary induced part. Note that the function  $\mathcal{I}_\mu(\beta/z^D)$  is negative for Dirichlet boundary condition ( $\beta = 0$ ) and is positive for non-Dirichlet boundary conditions assuming that  $z^D \ll |\beta|$ .

Now let us consider the behavior of the vacuum energy-momentum tensor in the asymptotic region  $z^D/\eta \gg 1$ . In this limit the main contribution in the integral over  $x$  in (39) comes from the region of the integration near the lower limit. As for the field squared, we consider the cases of the real and imaginary values  $\nu$  separately. For real  $\nu > 0$ , by using the asymptotic formulae for the modified Bessel functions for small values of the argument, in the leading order we find (no summation over  $i$ )

$$\langle T_i^i \rangle_{\text{pl}} \approx -\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \frac{2^{2\nu} \delta_\beta \Gamma((D+1)/2 - 2\nu) F_\nu^{(i)}}{(2\pi)^{(D+1)/2} \alpha^{D+1} (2z^D/\eta)^{D-2\nu}}, \quad (46)$$

for the diagonal components and

$$\langle T_0^D \rangle_{\text{pl}} \approx \frac{2^{2\nu} \Gamma(\nu)}{\Gamma(1-\nu)} \frac{\delta_\beta \Gamma((D+3)/2 - 2\nu) F_\nu^{(D)}}{(2\pi)^{(D+1)/2} \alpha^{D+1} (2z^D/\eta)^{D+1-2\nu}}, \quad (47)$$

for the off-diagonal component with  $\delta_\beta$  defined after formula (32). In these expressions the notations

$$\begin{aligned} F_\nu^{(0)} &= -D\xi + \frac{D-2\nu}{4} - \frac{m^2 \alpha^2}{2\nu}, \\ F_\nu^{(l)} &= \xi(D-2\nu+1) - \frac{D-2\nu}{4}, \quad l = 1, \dots, D, \end{aligned} \quad (48)$$

are introduced. As we see, in the limit under consideration the vacuum stresses are isotropic and  $|\langle T_0^D \rangle_{\text{pl}}| \ll |\langle T_0^0 \rangle_{\text{pl}}|$ . For  $\nu = 0$  one has the asymptotic expressions

$$\langle T_0^0 \rangle_{\text{pl}} \approx -\frac{2Dz^D}{\eta(D+1)} \langle T_0^D \rangle_{\text{pl}} \approx m^2 \alpha^2 \frac{2\delta_\beta \Gamma((D+1)/2) \ln(2z^D/\eta)}{(2\pi)^{(D+1)/2} \alpha^{D+1} (2z^D/\eta)^D}, \quad (49)$$

and for the diagonal stresses we have (no summation over  $i$ )  $|\langle T_i^i \rangle_{\text{pl}}| \ll |\langle T_0^0 \rangle_{\text{pl}}|$ ,  $i = 1, \dots, D$ .

For imaginary values  $\nu$  we use the asymptotic relation (33). To the leading order this gives the results (no summation over  $l$ )

$$\begin{aligned} \langle T_l^l \rangle_{\text{pl}} &\approx -\frac{\delta_\beta (2\pi)^{-(D+1)/2}}{(2z^D/\eta)^D \alpha^{D+1}} \frac{2|\nu| \pi B^{(l)}}{\sinh(|\nu|\pi)} \sin[2|\nu| \ln(2z^D/\eta) + \phi_l], \\ \langle T_0^D \rangle_{\text{pl}} &\approx \frac{2\delta_\beta (2\pi)^{-(D+1)/2}}{(2z^D/\eta)^{D+1} \alpha^{D+1}} \frac{\pi B^{(0D)}}{\sinh(|\nu|\pi)} \sin[2|\nu| \ln(2z^D/\eta) + \phi_{0D}], \end{aligned} \quad (50)$$

where the constants are defined by the relations

$$\begin{aligned} B^{(l)} e^{i\phi_l} &= 2^{2i|\nu|} \frac{i\Gamma((D+1)/2 - 2i|\nu|)}{\Gamma^2(1 - i|\nu|)} F_{i|\nu|}^{(l)}, \\ B^{(0D)} e^{i\phi_{0D}} &= 2^{2i|\nu|} \frac{\Gamma((D+3)/2 - 2i|\nu|)}{\Gamma^2(1 - i|\nu|)} F_{i|\nu|}^{(D)}. \end{aligned} \quad (51)$$

In this case the damping of the Casimir densities is oscillatory. For a fixed value of  $z^D$  the oscillating period in terms of the synchronous time coordinate is equal to  $\alpha/(2\pi)$ .

## 5 Dirichlet and Neumann boundary conditions

The formulae given above for the boundary induced parts in the VEVs of the field squared and the energy-momentum tensor are further simplified in the special cases of Dirichlet and Neumann boundary conditions. By using the formula

$$\int_y^\infty dx (x^2 - y^2)^{(D-3)/2} e^{-2z^D x/\eta} = 2^{D/2-1} \frac{\Gamma((D-1)/2)}{y^{2-D} \sqrt{\pi}} \frac{K_{D/2-1}(2z^D y/\eta)}{(2z^D y/\eta)^{D/2-1}}, \quad (52)$$

for the VEV of the field squared from (27) one finds

$$\langle \varphi^2 \rangle_{\text{pl}}^{(J)} = -\frac{2\delta^{(J)} \alpha^{1-D}}{(2\pi)^{D/2+1}} \int_0^\infty dy y^{D/2} [I_\nu(y) + I_{-\nu}(y)] K_\nu(y) \frac{K_{D/2-1}(2z^D y/\eta)}{(2z^D y/\eta)^{D/2-1}}, \quad (53)$$

with  $J=D, N$  for Dirichlet and Neumann boundary conditions and the definition for  $\delta^{(J)}$  is given after formula (25).

In the similar way, for the off-diagonal component of the VEV of the energy-momentum tensor we have

$$\langle T_0^D \rangle_{\text{pl}}^{(J)} = \frac{4\delta^{(J)} \alpha^{-D-1}}{(2\pi)^{D/2+1}} \int_0^\infty dy \frac{K_{D/2}(2z^D y/\eta)}{(2z^D y/\eta)^{D/2-1}} [(1/4 - \xi) y \partial_y - \xi] \tilde{I}_\nu(y) \tilde{K}_\nu(y). \quad (54)$$

For the diagonal components we find the expressions (no summation over  $i$ )

$$\begin{aligned} \langle T_i^i \rangle_{\text{pl}}^{(J)} = & -\frac{2\delta^{(J)} \alpha^{-D-1}}{(2\pi)^{D/2+1}} \int_0^\infty dy y \left\{ \frac{F_i^i(y)}{y^2} \frac{K_{D/2-1}(z)}{z^{D/2-1}} \right. \\ & \left. + F_i^i \tilde{I}_\nu(y) \tilde{K}_\nu(y) \left[ \frac{K_{D/2+1}(z)}{z^{D/2-1}} - \frac{K_{D/2}(z)}{z^{D/2}} \right] \right\}_{z=2z^D y/\eta}, \end{aligned} \quad (55)$$

with the notations (41). The corresponding asymptotic expressions are directly obtained from those given above for the general Robin case.

In figures 1 and 2 for conformally and minimally coupled  $D = 3$  Dirichlet scalar fields we have plotted the plate induced parts in the VEVs of the energy density and the energy flux along the direction normal to the plate as functions of the ratio  $z^D/\eta$ . Recall that the latter is the proper distance from the plate measured in units of the dS curvature scale  $\alpha$ . The numbers near the curves correspond to the values of the parameter  $m\alpha$ . We have taken these values in the way to have both real and imaginary values for the parameter  $\nu$ . In the first case (graphs for  $\xi = 1/6$ ,  $m\alpha = 1/4$  and  $\xi = 0$ ,  $m\alpha = 3/2$ ) the VEVs decay at large distances monotonically, whereas in the second case (graphs for  $\xi = 1/6$ ,  $m\alpha = 1$  and  $\xi = 0$ ,  $m\alpha = 2$ ) the corresponding behavior is damping oscillatory (see formulae (46), (47), (50) for the corresponding asymptotics). In the case of a conformally coupled field with  $m\alpha = 1$  the first zero is at  $z^D/\eta \approx 7.19$  for the energy density and at  $z^D/\eta \approx 9.64$  for the energy flux. For a minimally coupled field with  $m\alpha = 2$  the first two zeroes are located at  $z^D/\eta \approx 2.82, 8.67$  for the energy density and at  $z^D/\eta \approx 3.5, 11.3$  for the energy flux. The energy flux near the plate is positive for both conformally and minimally coupled scalars. For Neumann boundary conditions the VEVs have opposite signs.

It is also of interest to see the dependence of the boundary induced parts in the VEVs on the mass of the field. In figure 3 we have presented this dependence for the energy density and flux in the case of a conformally coupled field for the fixed distance from the plate corresponding to  $z^D/\eta = 3$ . For large values of the mass the corresponding asymptotic behavior is described by formulae (50).

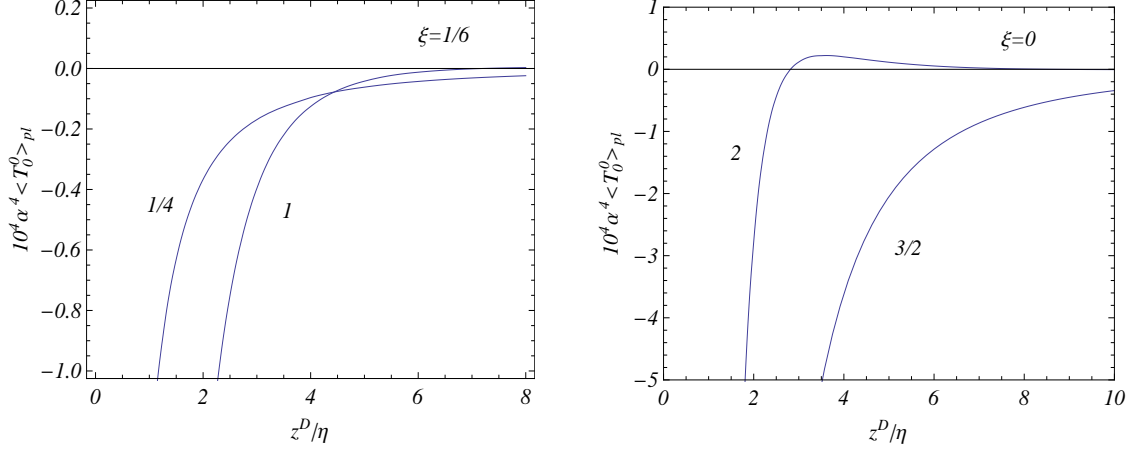


Figure 1: The boundary induced part in the VEV of the energy density for  $D = 3$  scalar field with Dirichlet boundary condition as a function of the ratio  $z^D/\eta$ . The left/right panel corresponds to fields with conformal/minimal coupling. The numbers near the curves are the values of the parameter  $\alpha m$ .

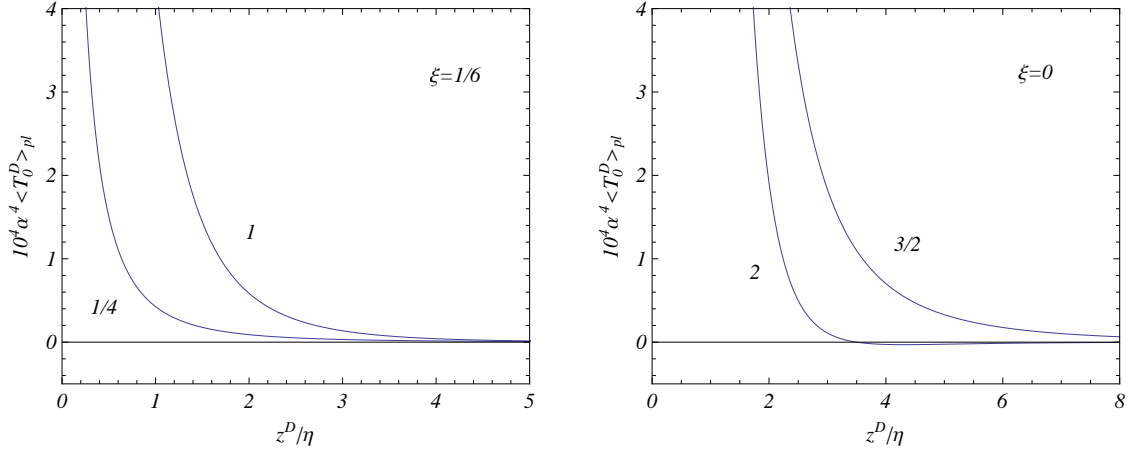


Figure 2: The same as in figure 1 for the energy flux.

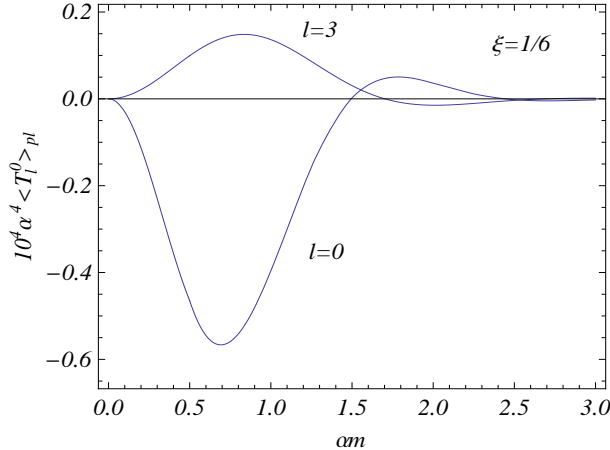


Figure 3: The boundary induced VEVs of the energy density and flux versus  $\alpha m$  in the case of  $D = 3$  conformally coupled scalar field with Dirichlet boundary condition for the fixed distance from the plate  $z^D/\eta = 3$ .

## 6 Conclusion

In the present paper we have investigated the VEVs of the field squared and the energy-momentum tensor for a massive scalar field with general curvature coupling parameter induced by a plane boundary in dS spacetime. The Robin boundary condition was assumed on the boundary. As the first step we have constructed the corresponding positive frequency Wightman function. This function is presented in the form of the sum of boundary-free and boundary induced parts. For points away from the boundary the latter is finite in the coincidence limit of the arguments and can be directly used for the evaluation of the VEVs of the field squared and the energy-momentum tensor. These VEVs are decomposed into boundary-free dS and plate induced parts. Due to the maximal symmetry of dS spacetime and the dS invariance of the Bunch-Davies vacuum the boundary-free parts do not depend on the spacetime point. These parts are well investigated in the literature and we were focused on the plate induced parts. The latter are given by formula (27) for the field squared and by formula (39) for the energy-momentum tensor and they are functions on the combinations  $z^D/\eta$  and  $\beta/\eta$  only. The first of these ratios is the proper distance from the plate measured in units of the curvature scale  $\alpha$ . An interesting feature is that the vacuum energy-momentum tensor is non-diagonal with the off-diagonal component corresponding to the energy flux along the direction normal to the plate. In dependence of the parameters, this flux can be either positive or negative. For special case of a conformally coupled massless scalar field the general formula for the VEV of the field squared is simplified to (30) and the plate induced energy-momentum tensor vanishes. These results can also be obtained from the corresponding flat spacetime results by using the conformal relation between the geometries.

For general values of the curvature coupling parameter the formulae for the plate induced VEVs are further simplified in the asymptotic regions of small and large proper distances from the plate. In the first case the leading term in the asymptotic expansions have the form (31) for the field squared and the form (45) for the energy-momentum tensor. Near the plate we have  $|\langle T_D^D \rangle_{\text{pl}}| \ll |\langle T_D^0 \rangle_{\text{pl}}| \ll |\langle T_0^0 \rangle_{\text{pl}}|$  and the total VEV of the energy-momentum tensor is dominated by the boundary induced part. At large proper distances from the plate the behavior of the plate induced parts is qualitatively different for real and pure imaginary values of the parameter

$\nu$ . In the first case these parts decay monotonically as  $(z^D/\eta)^{2\nu-D}$  for the field squared and for the diagonal components of the energy-momentum tensor and like  $(z^D/\eta)^{2\nu-D-1}$  for the off-diagonal component  $\langle T_0^D \rangle_{\text{pl}}$ . In this limit the diagonal vacuum stresses are isotropic. For imaginary values of the parameter  $\nu$  the asymptotic behavior of the plate induced VEVs at large proper distances from the plate is described by formulae (34) and (50) and is damping oscillatory. For a fixed value of  $z^D$  the oscillating period in terms of the synchronous time coordinate is equal to  $\alpha/(2\pi)$ .

The special cases of Dirichlet and Neumann boundary conditions are considered in section 5. For these cases the formulae for the expectation values are further simplified to (53)-(55) and they have opposite signs for Dirichlet and Neumann scalars.

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